

EXISTENCE, STABILITY AND OSCILLATION PROPERTIES OF SLOW DECAY POSITIVE SOLUTIONS OF SUPERCRITICAL ELLIPTIC EQUATIONS WITH HARDY POTENTIAL

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ABSTRACT. We prove the existence of a family of slow decay positive solutions of a supercritical elliptic equation with Hardy potential in \mathbb{R}^N and study stability and oscillation properties of these solutions. We also establish the existence of a continuum of stable slow decay positive solutions for the relevant exterior Dirichlet problem.

1. INTRODUCTION.

Our starting point is a superlinear elliptic problem in the entire space

$$(1.1) \quad -\Delta u = u^p, \quad u > 0 \quad \text{in } \mathbb{R}^N,$$

where $p > 1$ and $N \geq 3$. By $p_S := \frac{N+2}{N-2}$ in what follows we denote the *critical Sobolev exponent*. It is well-known that for $p < p_S$ problem (1.1) has no positive solutions. For finite energy solutions this is an easy consequence of Pohozaev's identity. For positive solutions without decay assumptions at infinity this is a deep result of Gidas and Spruck [6]. For $p = p_S$ all positive solutions of (1.1) are given up to translations by a one-parameter family

$$W_\lambda(|x|) := \lambda^{\frac{N-2}{2}} W_1(\lambda|x|) \quad (\lambda > 0),$$

where $W_1(x) := (1 + (N(N-2))^{-1}|x|^2)^{-\frac{N-2}{2}}$ is a rescaled minimizer of the Sobolev inequality.

For $p > p_S$ the structure of the solution set of (1.1) is more complex. First we note that for all $p > \frac{N}{N-2}$ problem (1.1) possesses an explicit *singular* radial positive solution

$$U_\infty(x) := C_p |x|^{-\frac{2}{p-1}}, \quad C_p := \left(\frac{2}{p-1} \left(N-2 - \frac{2}{p-1} \right) \right)^{\frac{1}{p-1}}.$$

Observe that if $p > p_S$ then $U_\infty \in H_{loc}^1(\mathbb{R}^N)$ and hence U_∞ is a weak solution of (1.1) in the entire \mathbb{R}^N , despite the singularity at the origin. However U_∞ is an infinite energy solution because of its slow decay at infinity for $p > p_S$.

The set of all radially symmetric solutions of (1.1) can be analyzed through phase plane analysis after applying Fowler's transformation, cf. [14, p. 50-55]. In particular, if $p > p_S$ then (1.1) admits a radial positive solution $U_1(|x|)$ such that $U_1(0) = 1$. It is known that $U_1(|x|)$ is monotone decreasing and

$$\lim_{|x| \rightarrow \infty} \frac{U_1(|x|)}{|x|^{-\frac{2}{p-1}}} = C_p,$$

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however U_1 has no explicit representation in terms of elementary functions. Taking into account the scaling invariance one concludes that rescalings of U_1 are also solutions of (1.1), so that (1.1) possess a one-parameter continuum of radial positive solutions

$$(1.2) \quad U_\lambda(|x|) = \lambda^{\frac{2}{p-1}} U_1(\lambda|x|) \quad (\lambda > 0).$$

One can show that the singular solution U_∞ is the limit of the family (U_λ) , in the sense that for any $x \neq 0$ holds

$$\lim_{\lambda \rightarrow \infty} U_\lambda(|x|) = U_\infty(|x|).$$

In addition, it is known that given $0 < \lambda_1 < \lambda_2 \leq \infty$, solutions $U_{\lambda_1}(r)$ and $U_{\lambda_2}(r)$ in the range $p_S < p < p_{JL}$ intersect each other infinitely many times as $r \rightarrow \infty$, while for $p \geq p_{JL}$ the solutions are strictly ordered, that is $U_{\lambda_1}(r) < U_{\lambda_2}(r)$ for all $r \geq 0$. Here

$$p_{JL} := \begin{cases} \frac{N-2\sqrt{N-1}}{N-4-2\sqrt{N-1}}, & \text{if } N > 10, \\ \infty & \text{if } N \leq 10, \end{cases}$$

is the *Joseph–Lundgren stability exponent*, introduced in [10]. The exponent p_{JL} controls various oscillation and stability properties of solutions U_λ , which are particularly important in the study of the time-dependent parabolic version of (1.1), see [8, 17] or [14, p. 50-55] for a discussion.

We are interested in a perturbation of (1.1) by the Hardy inverse square potential, that is the equation

$$(1.3) \quad -\Delta u + \frac{\mu}{|x|^2} u = u^p, \quad u > 0 \quad \text{in } \mathbb{R}^N \setminus \{0\},$$

where $\mu > -C_H$ and $C_H := \frac{(N-2)^2}{4}$ is the *Hardy critical constant*, i.e. the optimal constant in the Hardy inequality

$$(1.4) \quad \int_{\mathbb{R}^N} |\nabla \varphi|^2 dx \geq C_H \int_{\mathbb{R}^N} \frac{|\varphi|^2}{|x|^2} dx \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N).$$

Hardy potential provides an important example of a long range potential, that is a potential which modifies asymptotic decay rate of solutions at infinity and their behavior at the origin, see e.g. [2, 7].

For $p \neq p_S$ a Pohozaev-type identity shows that similarly to (1.1), equation (1.3) has no finite energy solutions [16]. For $p = p_S$ equation (1.3) admits an explicit one-parameter family of finite energy radial solutions, cf. [2, 16]. However, the structure of positive solutions of (1.3) in the *critical regime* $p = p_S$ is not fully understood. It is known that for large values of $\mu > 0$ equation (1.3) admits nonradial solutions which are distinct modulo rescalings from the radial solutions [2, 16]. See [9] for recent results and discussion of open questions in this direction.

In the present work we consider equation (1.3) in the *supercritical regime* $p > p_S$. In the next section we setup the problem and discuss basic properties of the explicit singular solution similar to U_∞ . In Section 3, for optimal ranges of p and μ we establish the existence of a one-parameter family $(U_\lambda)_{\lambda>0}$ of infinite energy solutions of (1.3), which coincides with (1.2) when $\mu = 0$. We also discuss stability properties of these solutions. The presence of the Hardy potential produces a range of new critical exponents related to stability which do not have immediate analogues in the unperturbed case of equation (1.1). Finally in Section 4, we discuss equation (1.3) in exterior domains. We justify optimality of critical exponents introduced in previous sections. Further, under some assumptions on p and μ we prove the existence of a continuum of infinite energy solutions of (1.3), which in some sense

could be considered as a perturbation of the original family of solutions on \mathbb{R}^N but goes beyond spherically symmetric or scaling invariant setting. This partially extends some of the recent results in [4].

2. EQUATIONS WITH HARDY POTENTIAL.

We study the equation

$$(2.1) \quad -\Delta u + \frac{\nu^2 - \nu_*^2}{|x|^2} u = u^p, \quad u > 0 \quad \text{in } \mathbb{R}^N \setminus K,$$

where $K = \{0\}$, or $\{0\} \in K$ and K is a connected compact set with the smooth boundary ∂K , $p > 1$, $N \geq 3$, $\nu > 0$ and $\nu_* := \frac{N-2}{2}$, so that ν_*^2 is the Hardy critical constant in (1.4). By a solution of (2.1) we understand a classical solution $u \in C^2(\mathbb{R}^N \setminus K)$, with no apriori assumption on the decay of $u(x)$ at infinity. We say u is a weak solution of (2.1) in \mathbb{R}^N if $u \in H_{loc}^1(\mathbb{R}^N)$ and

$$\int \nabla u \cdot \nabla \varphi \, dx + (\nu^2 - \nu_*^2) \int \frac{u\varphi}{|x|^2} \, dx = \int u^p \varphi \, dx \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N).$$

Note that for $\nu < \nu_*$ solutions of (2.1) must have a singularity at the origin (see Lemma 2.5 below) however this singularity might be compatible with the concept of a weak solution in \mathbb{R}^N .

We say a solution u of (2.1) in $\mathbb{R}^N \setminus K$ has *finite energy* if $u \in D^1(\mathbb{R}^N \setminus K)$, the completion of $C_c^\infty(\mathbb{R}^N \setminus K)$ with respect to the norm $\|\nabla \varphi\|_{L^2}$. We say a solution u of (2.1) is *stable* in $\mathbb{R}^N \setminus K$ if the formal second variation at u of the energy which corresponds to (2.1) is nonnegative definite, that is

$$(2.2) \quad \int |\nabla \varphi|^2 \, dx + (\nu^2 - \nu_*^2) \int \frac{\varphi^2}{|x|^2} \, dx - p \int u^{p-1} \varphi^2 \, dx \geq 0 \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N \setminus K).$$

A solution $u > 0$ of (2.1) is called *semi-stable* in $\mathbb{R}^N \setminus K$ if it is stable in $\mathbb{R}^N \setminus B_R$, for some $R > 0$. A solution $u > 0$ of (2.1) is called *unstable* if it is not semistable. Note that these definitions do not require u to be a finite energy solution.

2.1. Explicit radial solution. For $\nu > 0$ and $p > p_* := 1 + \frac{2}{\nu_* + \nu}$, set

$$U_\infty(x) := C_{p,\nu} |x|^{-\frac{2}{p-1}}, \quad C_{p,\nu}^{p-1} := \nu^2 - \left(\nu_* - \frac{2}{p-1} \right)^2,$$

and introduce the critical exponent

$$p^* := \begin{cases} 1 + \frac{2}{\nu_* - \nu}, & \text{if } \nu < \nu_*, \\ \infty & \text{if } \nu \geq \nu_*, \end{cases}$$

Clearly $p^* > p_S$. A direct computation shows that U_∞ is a positive solution of (2.1) for all $p_* < p < p^*$, while for $p \notin [p_*, p^*]$ the coefficient $C_{p,\nu}$ becomes negative. Note that $U_\infty \in H_{loc}^1(\mathbb{R}^N)$ for $p > p_S$, that is U_∞ is a weak solution of (2.1) in \mathbb{R}^N . However U_∞ is an infinite energy solution because of its slow decay at infinity.

The importance of the solution U_∞ is due to the fact that it will be used as an elementary building block for constructing further solutions of (2.1). In order to do this it is essential to understand stability properties of U_∞ .

Lemma 2.1. *Let $p \in (p_*, p^*)$ and $\nu > 0$. The solution U_∞ is stable if and only if*

$$(2.3) \quad p C_{p,\nu}^{p-1} \leq \nu^2,$$

while if (2.3) fails then U_∞ is unstable.

Proof. The formal second variation of the energy which corresponds to (2.1) at U_∞ is given by

$$\int |\nabla \varphi|^2 dx + (\nu^2 - \nu_*^2 - pC_{p,\nu}^{p-1}) \int \frac{|\varphi|^2}{|x|^2} dx.$$

Thus the assertion follows directly from the fact that ν_*^2 is the optimal constant in the Hardy inequality (1.4).

Taking into account the scaling invariance of Hardy's inequality we also conclude that if (2.3) fails then U_∞ must be unstable. \square

The inequality (2.3) amounts to a third degree algebraic expression for which closed form solutions could be obtained using Cardano's formulae, however the explicit expressions for solutions are tedious. Below we present a qualitative analysis of (2.3). Set $s := -\frac{2}{p-1}$, so that (2.3) transforms into

$$\frac{(s + \nu_*)^2(s - 2) + 2\nu^2}{|s|} \leq 0 \quad (-\nu_* - \nu < s < \min\{-\nu_* + \nu, 0\}).$$

Define

$$\theta(s) := (s + \nu_*)^2(s - 2).$$

Then solving (2.3) for $p_* < p < p^*$ is equivalent to classifying the roots of the equation

$$(2.4) \quad \theta(s) = -2\nu^2 \quad (-\nu_* - \nu < s < \min\{-\nu_* + \nu, 0\}),$$

and solving the inequality

$$(2.5) \quad \theta(s) \leq -2\nu^2 \quad (-\nu_* - \nu < s < \min\{-\nu_* + \nu, 0\}).$$

Note that $\theta(0) = -2\nu_*^2$ and that θ has two critical points: a local maximum at $s_{max} := -\nu_*$ with $\theta(s_{max}) = 0$ and a local minimum at $s_{min} := -\frac{\nu_*-4}{3}$ with $\theta(s_{min}) = -\frac{4}{27}(2 + \nu_*)^3$. Denote

$$\bar{\nu} := \sqrt{\frac{2}{27}(2 + \nu_*)^3} = \sqrt{2 \left(\frac{N+2}{6} \right)^3}.$$

Clearly for every $\nu > 0$ equation (2.4) has exactly one root $\sigma_\#$ in the interval $(-\nu_* - \nu, -\nu_*)$. To analyze the roots of (2.5) in the interval $(-\nu_*, \min\{-\nu_* + \nu, 0\})$ we distinguish the cases $s_{min} < 0$ and $s_{min} \geq 0$.¹

In the case $s_{min} \geq 0$ (that is $3 \leq N \leq 10$):

(i) if $\nu \geq \nu_*$ then (2.4) has no roots in $(-\nu_*, 0)$ and (2.5) holds $\forall s \in (-\nu_* - \nu, \sigma_\#]$,

(ii) if $0 < \nu < \nu_*$ then (2.4) has exactly one root $\sigma_- \in (-\nu_*, -\nu_* + \nu)$ and (2.5) holds $\forall s \in (-\nu_* - \nu, \sigma_\#] \cup [\sigma_-, -\nu_* + \nu)$.

In the case $s_{min} < 0$ (that is $N > 10$):

(i) if $\nu > \bar{\nu}$ then (2.4) has no roots in $(-\nu_*, 0)$ so that (2.5) holds $\forall s \in (-\nu_* - \nu, \sigma_\#]$,

(ii) if $\nu_* < \nu \leq \bar{\nu}$ then (2.4) has exactly 2 roots σ_- and σ_+ in $(-\nu_*, 0)$ and $-\nu_* < \sigma_- \leq s_{min} \leq \sigma_+ < 0$ so that (2.5) holds $\forall s \in (-\nu_* - \nu, \sigma_\#] \cup [\sigma_-, \sigma_+]$,

(iii) if $0 < \nu \leq \nu_*$ then (2.4) has exactly 1 root σ_- in $(-\nu_*, 0)$ and $\sigma_- \in (-\nu_*, s_{min})$ so that (2.5) holds $\forall s \in (-\nu_* - \nu, \sigma_\#] \cup [\sigma_-, 0)$.

In what follows we denote

$$p_\# = 1 - \frac{2}{\sigma_\#}, \quad p_- := 1 - \frac{2}{\sigma_-}, \quad p_+ := 1 - \frac{2}{\sigma_+},$$

¹Note that if we write $\bar{\mu} = \bar{\nu}^2 - \nu_*^2$ as in (1.3) then $\bar{\mu} = \frac{1}{108}(N-10)^2(N-1)$.

and note that

$$1 < p_* < p_{\#} < p_S < p_- \leq p_+ < p^*,$$

for all values of $N \geq 3$ and $\nu > 0$ when all the exponents are well defined. Then the above analysis leads to the following equivalent to (2.3) characterization of the stability properties of the solution U_{∞} in terms of the original parameters p and ν .

Lemma 2.2. *Let $p \in (p_*, p^*)$ and $\nu > 0$.*

- (a) *If $\nu_* < \nu \leq \bar{\nu}$ and $N \geq 11$ then the solution U_{∞} is stable for $p \in (p_*, p_{\#}] \cup [p_-, p_+]$ and unstable for $p \in (p_{\#}, p_-) \cup (p_+, p^*)$.*
- (b) *If $0 < \nu < \nu_*$ and $N \geq 3$ or $\nu = \nu_*$ and $N \geq 11$ then the solution U_{∞} is stable for $p \in (p_*, p_{\#}] \cup [p_-, p^*)$ and unstable for $p \in (p_{\#}, p_-)$.*
- (c) *If $\nu \geq \nu_*$ and $3 \leq N \leq 10$ or $\nu \geq \bar{\nu}$ and $N \geq 11$ the solution U_{∞} is stable for $p \in (p_*, p_{\#}]$ and unstable for $p \in (p_{\#}, \infty)$.*

Remark 2.3. In the pure Laplacian case $\nu = \nu_*$ one calculates the explicit values

$$p_{\#} = \frac{N + 2\sqrt{N-1}}{N - 4 + 2\sqrt{N-1}}, \quad p_- = \frac{N - 2\sqrt{N-1}}{N - 4 - 2\sqrt{N-1}},$$

here p_- is defined only for $N \geq 11$. Thus for the Laplacian the exponent p_- coincides with the Joseph–Lundgren stability exponent, see [10] or [14, p.50]; while the exponent $p_{\#}$ is known to appear in the context of local singularities of solution of equations (1.1), cf. [13, Lemma 5].

Remark 2.4. If $N > 10$ and $\nu = \bar{\nu}$ then $p_- = p_+ = \frac{N+2}{N-10}$ is the only supercritical value of p where U_{∞} is stable.

2.2. Slow and fast decay solutions. Clearly, a solution u of (2.1) is a positive superharmonic of the linear Hardy operator, that is u satisfy the linear inequation

$$(2.6) \quad -\Delta u + \frac{\nu^2 - \nu_*^2}{|x|^2} u \geq 0 \quad \text{in } \mathbb{R}^N \setminus K.$$

As a consequence, solutions of (2.1) with $\nu^2 < \nu_*^2$ are always singular at the origin while for $\nu^2 > \nu_*^2$ solutions might vanish at the origin. More precisely, the following local decay properties for positive superharmonics of Hardy's operator hold, cf. [12].

Lemma 2.5. *If $u > 0$ satisfy (2.6) in a neighborhood of the origin then*

$$(2.7) \quad \liminf_{|x| \rightarrow 0} \frac{u(x)}{|x|^{-\nu_* + \nu}} > 0, \quad \liminf_{|x| \rightarrow 0} \frac{u(x)}{|x|^{-\nu_* - \nu}} < \infty.$$

If $u > 0$ satisfy (2.6) in an exterior domain then

$$(2.8) \quad \liminf_{|x| \rightarrow \infty} \frac{u(x)}{|x|^{-\nu_* - \nu}} > 0, \quad \liminf_{|x| \rightarrow 0} \frac{u(x)}{|x|^{-\nu_* + \nu}} < \infty.$$

Bidaut–Véron and Véron [2, Theorem 3.3] proved that the structure of the solution set of (2.1) in exterior domains which decay at infinity no slower than U_{∞} is essentially determined by the solutions of the following equation

$$(2.9) \quad -\Delta_{S^{N-1}} \omega + C_{p,\nu}^{p-1} \omega = \omega^p, \quad \omega > 0 \quad \text{in } S^{N-1}.$$

on the sphere S^{N-1} .

Lemma 2.6. [2, Theorem 3.3] *Let $p \neq p_S$. If $u > 0$ satisfy (2.1) in $\mathbb{R}^N \setminus K$ and*

$$(2.10) \quad \limsup_{|x| \rightarrow \infty} \frac{u(x)}{|x|^{-\frac{2}{p-1}}} < \infty,$$

then either

$$(2.11) \quad \lim_{|x| \rightarrow \infty} \frac{u(x)}{|x|^{-\nu_* - \nu}} = c \quad (\text{fast decay}),$$

or there exists a positive solution $\omega(\cdot)$ of (2.9) such that

$$(2.12) \quad \lim_{|x| \rightarrow \infty} \frac{u(|x|, \cdot)}{|x|^{-\frac{2}{p-1}}} = \omega(\cdot) \quad (\text{slow decay})$$

in the $C^k(S^{N-1})$ topology, for any $k \in \mathbb{N}$.

Remark 2.7. Clearly, $C_{p,\nu}$ is a constant solution of (2.9). For $1 < p < \frac{N+1}{N-3}$ it is known (see [6] or [2, Corollary 6.1]) that $C_{p,\nu}$ is the only solution of (2.9) provided that

$$(2.13) \quad (p-1)C_{p,\nu}^{p-1} \leq N-1,$$

while if (2.13) fails then problem (2.9) admits nonconstant solutions, see [2, Corollary 6.1], [16, Theorem 0.5] and [9, Theorem 1.3]. Similar result holds for some values $p > \frac{N+1}{N-3}$, see [3]. The complete structure of solution set of (2.9) is not yet fully understood, see [9, 1] for some recent results in this direction.

Remark 2.8. If $\nu > 0$ and $p < p_S$ then (2.10) always holds, see [2, Remark 3.2].

We will classify positive solutions of (2.1) into *fast* and *slow decay solutions* according to alternatives (2.11) and (2.12). Note that for $p > p_S$ slow decay solutions are always infinite energy solutions, because of the slow decay rate (2.12) at infinity.

3. RADIAL SLOW DECAY SOLUTIONS IN \mathbb{R}^N .

Radial positive solutions $u(|x|) > 0$ of (2.1) in $\mathbb{R}^N \setminus \{0\}$ correspond to the positive solutions $U(r) = u(r)$ of the initial value problem

$$(3.1) \quad -U'' - \frac{N-1}{r}U' + \frac{\nu - \nu_*}{r^2}U = U^p \quad (r > 0),$$

which can be studied through the phase plane analysis.

The existence of a family of regular at the origin slow-decay solutions of (3.1) in the Laplacian case $\nu = \nu_*$ is well-known and goes back at least to [10].

Theorem 3.1. *Let $p_S < p < p^*$. Then for any $\lambda > 0$ equation (3.1) admits a unique positive solution $U_\lambda \in C^2(0, \infty)$ such that*

$$(3.2) \quad \lim_{r \rightarrow 0} \frac{U_\lambda(r)}{r^{-\nu_* + \nu}} = \lambda, \quad \lim_{r \rightarrow \infty} \frac{U_\lambda(r)}{r^{-\frac{2}{p-1}}} = C_{p,\nu}.$$

Moreover,

$$(3.3) \quad U_\lambda(r) = \lambda^{\frac{2}{p-1}} U_1(\lambda r) \quad \forall \lambda > 0.$$

Further, for $\lambda \in (0, \infty]$ the following properties hold:

- (i) if $pC_{p,\nu}^{p-1} \leq \nu^2$ then solutions U_λ are stable and ordered in the sense that $0 < \lambda_1 < \lambda_2 \leq \infty$ implies $U_{\lambda_1}(r) < U_{\lambda_2}(r)$ for every $r \geq 0$ and in addition,

$$(3.4) \quad \lim_{r \rightarrow \infty} \frac{U_{\lambda_2}(r) - U_{\lambda_1}(r)}{r^{-\nu_*}} > 0;$$

- (ii) if $pC_{p,\nu}^{p-1} > \nu^2$ then solutions U_λ unstable and oscillate, in the sense that $0 < \lambda_1 < \lambda_2 \leq \infty$ implies that $U_{\lambda_2}(r) - U_{\lambda_1}(r)$ changes sign in $(R, +\infty)$ for arbitrary $R > 0$.

The proof of the theorem follows the exposition in [14, pp.50-53] with minor adjustments needed to accommodate $\nu \neq \nu_*$. We present the sketch of the arguments for the readers convenience.

Proof of Theorem 3.1. Using the transformation

$$(3.5) \quad w(t) = r^{\frac{2}{p-1}} U(r), \quad t = \log(r),$$

problem (3.1) becomes an autonomous second order differential equation

$$(3.6) \quad w'' + 2\beta w' + w^p - \gamma w = 0, \quad t \in \mathbb{R},$$

where since $p_S < p < p^*$,

$$\beta := \nu_* - \frac{2}{p-1} > 0, \quad \text{and} \quad \gamma = C_{p,\nu}^{p-1} = \nu^2 - \left(\nu_* - \frac{2}{p-1} \right)^2 > 0.$$

Set

$$\mathcal{E}(w) = \mathcal{E}(w, w') := \frac{1}{2}|w'|^2 - \frac{\gamma}{2}w^2 + \frac{1}{p+1}w^{p+1}.$$

Then \mathcal{E} is a Lyapunov function for (3.6) and

$$\frac{d}{dt}\mathcal{E}(w(t)) = -2\beta(w'(t))^2 \leq 0.$$

Set $x := w$ and $y := w'$. Then (3.6) can be written as an autonomous first order system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} y \\ -2\beta y + \gamma x - x^p \end{pmatrix} =: \Phi(x, y),$$

which possesses two equilibria

$$(0, 0) \quad \text{and} \quad (\gamma^{\frac{1}{p-1}}, 0)$$

in the half-space $\{(x, y) : x \geq 0\}$. Denote

$$A_0 := \nabla \Phi(0, 0) = \begin{pmatrix} 0 & 1 \\ \gamma & -2\beta \end{pmatrix}, \quad A_* := \nabla \Phi(\gamma, 0) = \begin{pmatrix} 0 & 1 \\ -(p-1)\gamma & -2\beta \end{pmatrix}.$$

The matrix A_0 has two real eigenvalues

$$\alpha_{\pm} := -\beta \pm \sqrt{\beta^2 + \gamma} = \frac{2}{p-1} - \nu_* \pm \nu,$$

so that $\alpha_- < 0 < \alpha_+$. The corresponding eigenvectors are $(1, \alpha_+)$ and $(1, \alpha_-)$, that is $(0, 0)$ is a saddle point of the vector field Φ . The matrix A_* has two eigenvalues

$$\alpha_{\pm}^* := -\beta \pm \sqrt{\beta^2 - (p-1)\gamma},$$

the corresponding eigenvectors are $(1, \alpha_+^*)$ and $(1, \alpha_-^*)$. Clearly $\text{Re}(\alpha_{\pm}^*) < 0$, so $(\gamma^{\frac{1}{p-1}}, 0)$ is always an attractor. Note also that α_{\pm}^* is real if and only if $pC_{p,\nu}^{p-1} \leq \nu^2$.

Using the Lyapunov function \mathcal{E} one can show that the trajectory tangent at the origin to the eigenvector $(1, \alpha_+)$ is a heteroclinic orbit which connects the equilibria $(0, 0)$ and $(\gamma^{\frac{1}{p-1}}, 0)$, see [14, p.52]. Moreover, since $(0, 0)$ is a hyperbolic saddle-point, the uniqueness of such heteroclinic orbit follows by standard arguments. The corresponding solution $w(t)$ exists for all $t \in \mathbb{R}$ and satisfies

$$(3.7) \quad \lim_{t \rightarrow -\infty} w(t) = 0, \quad \lim_{t \rightarrow +\infty} w(t) = \gamma^{\frac{1}{p-1}}.$$

Moreover, we can assume that $w(t)$ satisfies the normalization condition

$$(3.8) \quad \lim_{t \rightarrow -\infty} \frac{w(t)}{e^{\alpha_+ t}} = 1.$$

Since (3.6) is autonomous, $w(t + \theta)$ is also a solution of (3.6) that corresponds to the same heteroclinic orbit, for any $\theta \in \mathbb{R}$. Given $\theta \in \mathbb{R}$, set $\lambda := e^\theta$. Then

$$U_\lambda(r) := r^{-\frac{2}{p-1}} w(\log(\lambda r)) = \lambda^{\frac{2}{p-1}} U(\lambda r),$$

and U_λ satisfies (3.2) in view of (3.8) and (3.7), that is U_λ is the required solution of (3.1). The uniqueness of U_λ follows from the uniqueness of $w(t)$ since (3.5) defines a one to one correspondence between solutions of (3.1) and (3.6).

To understand oscillation and stability properties of U_λ note that the eigenvalues α_\pm^* are real iff

$$\beta^2 \geq (p-1)\gamma,$$

which is equivalent to the stability condition (2.3). Note that then

$$\alpha_- < \alpha_-^* \leq -\nu_* \leq \alpha_+^* < \alpha_+.$$

If the roots α_\pm^* are real then arguments similar to [14, p.53] show that the trajectory $w(t)$ is monotone increasing in t for all $t \in \mathbb{R}$. Hence the solutions $U_\lambda(r)$ are monotone increasing in λ . In particular, $U_\lambda(r) < U_\infty(r)$ for any $\lambda > 0$ and solutions U_λ are ordered. Further, in view of (2.3) the solution U_∞ is stable. Since $U_\lambda(r) < U_\infty(r)$, we obtain

$$pU_\lambda^{p-1}(|x|) \leq pU_\infty^{p-1}(|x|) = p\gamma|x|^2 \leq \nu^2|x|^2.$$

By Hardy's inequality we conclude that

$$\int |\nabla \varphi|^2 dx + (\nu^2 - \nu_*^2) \int \frac{\varphi^2}{|x|^2} dx - p \int U_\lambda^{p-1}(|x|) \varphi^2 \geq 0$$

for all $\varphi \in C_c^\infty(\mathbb{R}^N)$, that is U_λ is a stable solution of (2.1). In addition, similarly to [14, Remark 9.4], we conclude that

$$\lim_{t \rightarrow \infty} \frac{w'(t)}{w(t) - \gamma^{\frac{1}{p-1}}} = \alpha_+^* \geq -\beta,$$

which after returning to the original variables and combined with (3.3) implies (3.4).

If α_\pm^* are complex then similarly to [14, p.52] one can see that the trajectory $(x(t), y(t))$ spirals infinitely many times around the attractor $(\gamma, 0)$ which suggests that the solutions U_λ oscillate in the sense of (ii). The detailed prove of oscillation and instability of U_λ when α_\pm^* are complex is a particular case of a more general Theorem 4.3 which will be proved in the next Section. \square

Remark 3.2. In the subcritical case $p_* < p \leq p_S$ equation (3.1) has no positive slow decay solution which satisfy (3.2). Indeed, if $p = p_S$ then $\beta = 0$, $\text{Re}(\alpha_\pm^*) = 0$ and the stationary point $(\gamma^{\frac{1}{p-1}}, 0)$ is a center. One can show that the trajectory tangent at the origin to the eigenvector $(1, \alpha_+)$ is a homoclinic orbit. This homoclinic corresponds to an explicit one parameter family of finite energy solutions of (3.1), see [16, pp.253-254]. If $p_* < p < p_S$ then $\beta > 0$, $\text{Re}(\alpha_\pm^*) > 0$ and the stationary point $(\gamma^{\frac{1}{p-1}}, 0)$ is repelling. Hence a heteroclinic between $(\gamma^{\frac{1}{p-1}}, 0)$ and $(0, 0)$ originates at $(\gamma^{\frac{1}{p-1}}, 0)$ and converges to $(0, 0)$ tangentially to the eigenvector $(1, \alpha_-)$. This heteroclinic corresponds to a positive solution of (3.1) which decays at infinity as $O(|x|^{-\nu_* - \nu})$ and has a singularity at the origin of order $O(|x|^{-\frac{2}{p-1}})$.

4. SLOW DECAY SOLUTIONS IN EXTERIOR DOMAINS.

First we justify that the value of the nonexistence exponent p^* is sharp. The result, which is first appeared in [2, Remark 3.2], is an immediate consequence of Lemma 2.6.

Theorem 4.1. *Let $p \geq p^*$. Then (2.1) has no slow decay solutions in $\mathbb{R}^N \setminus \bar{B}_R$, for arbitrary $R > 0$.*

Proof. Simply note that for $p > p^*$ one has $C_{p,\nu} \leq 0$ and hence the equation (2.9) on the sphere does not have any positive solution. Then the conclusion follows from Lemma 2.6. \square

Remark 4.2. If $p > p^*$ then the slow decay rate is incompatible with the upper bound (2.8) of Lemma 2.5. This argument however does not apply when $p = p^*$.

Next we justify sharpness of the stability and nonoscillation condition (2.3). The result below extends oscillation statement of Theorem 3.1 beyond radial setting. See also [17, Proposition 3.5] for related results in the pure Laplacian case $\nu = \nu_*$.

Theorem 4.3. *Let $p > p_S$, $\nu > 0$ and $pC_{p,\nu}^{p-1} > \nu^2$. Let $U_* > 0$ be a subsolution of (2.1) such that*

$$(4.1) \quad \liminf_{|x| \rightarrow \infty} \frac{U_*(x)}{|x|^{-\frac{2}{p-1}}} \geq C_{p,\nu}.$$

Then U_ is unstable. Further, if $u > 0$ is a supersolution of (1.1) then either $u = U_*$, or $(u - U_*)_+ \neq 0$ in $\mathbb{R}^N \setminus \bar{B}_R$, for arbitrary large $R > 0$.*

Proof. From (4.1) we obtain

$$pU_*^{p-1}(x) \geq (\nu^2 + \varepsilon)|x|^{-2} \quad (|x| > R_\varepsilon),$$

for some $\varepsilon > 0$ and $R_\varepsilon \geq R$. Assume that U_* is semistable, that is there exists $R > 0$ such that (2.2) holds in $\mathbb{R}^N \setminus \bar{B}_R$. But then we arrive at

$$\int |\nabla \varphi|^2 dx + (\nu^2 - \nu_*^2) \int \frac{\varphi^2}{|x|^2} dx \geq (\nu^2 + \varepsilon) \int \frac{\varphi^2}{|x|^2} dx \geq 0 \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N \setminus \bar{B}_{R_\varepsilon}),$$

a contradiction to Hardy's inequality. We conclude that U_* is unstable.

Further, set $h = u - U_*$ and assume that $h \geq 0$ in $\mathbb{R}^N \setminus \bar{B}_R$, for some $R > 1$. Then by convexity and (4.1) we obtain

$$\begin{aligned} -\Delta h + \frac{\nu^2 - \nu_*^2}{|x|^2} h &\geq u^p - U_*^p = (U_* + h)^p - U_*^p \\ &\geq pU_*^{p-1}h = \frac{pC_{p,\nu}^{p-1}}{|x|^2}h \geq \frac{\nu^2 + \varepsilon^2}{|x|^2}h \quad (|x| > R_\varepsilon). \end{aligned}$$

It is well-known that such inequation has no positive solutions, cf. [12, Corollary 3.2]. We conclude that either $h = 0$, or h changes sign in $\mathbb{R}^N \setminus \bar{B}_{R_\varepsilon}$. \square

Remark 4.4. The above result does not exclude possibility that $u < U_*$ in an exterior domain. The latter is however not possible in the case when both U_* and u are slow decay solutions. In particular, since all the solutions U_λ satisfy (4.1), the above result includes the oscillation statement (ii) of Theorem 3.1.

Next we show that if the stability assumption (2.3) holds then slow decay solutions of (2.1) in exterior domains are well ordered in a certain sense. We consider

the exterior boundary value problem for (2.1)

$$(4.2) \quad \begin{cases} -\Delta u + \frac{\nu^2 - \nu_*^2}{|x|^2} u = u^p, & u > 0 & \text{in } \mathbb{R}^N \setminus K, \\ u = \psi & & \text{on } \partial K, \end{cases}$$

here $K \ni \{0\}$ is a connected compact set with the smooth boundary ∂K , and $\psi \in C(\partial K)$ is a nonnegative continuous function.

Theorem 4.5. *Let $p > p_S$, $\nu > 0$ and $pC_{p,\nu}^{p-1} \leq \nu^2$. Let $U_* > 0$ be a slow decay solution of (2.1) in $\mathbb{R}^N \setminus K$ such that for some $R > 0$ holds*

$$U_*(x) \leq U_\infty(x) \quad (|x| > R).$$

Given $\psi \in C(\partial K)$ such that

$$0 \leq \psi(x) \leq U_*(x) \quad \text{on } \partial K,$$

problem (4.2) admits a slow decay solution U_^ψ such that*

$$0 < U_*^\psi \leq U_* \quad \text{in } \mathbb{R}^N \setminus K.$$

Moreover,

$$(4.3) \quad \lim_{|x| \rightarrow \infty} \frac{U_*(x) - U_*^\psi(x)}{|x|^{-\nu_*}} = 0.$$

Proof. We are going to construct a sub-solution \underline{U} and a super-solution \overline{U} such that

$$0 \leq \underline{U} \leq \overline{U} \leq U_* \quad \text{and} \quad \underline{U} = \overline{U} = \psi \quad \text{on } \partial K.$$

Then the existence of a solution U_*^ψ between \underline{U} and \overline{U} follows via the classical sub and super-solution argument, cf. [11, Theorem 38.1].

Subsolution \underline{U} . Let $h_\psi > 0$ be the minimal positive solution to the problem

$$(4.4) \quad -\Delta h + \frac{\nu^2 - \nu_*^2}{|x|^2} h = pU_*^{p-1} h \quad \text{in } \mathbb{R}^N \setminus K, \quad h = U_* - \psi \quad \text{on } \partial K.$$

The existence of such a solution is ensured by the Lax–Milgram theorem. Indeed, by the assumptions

$$(4.5) \quad pU_*^{p-1}(x) \leq pU_\infty^{p-1}(x) \leq pC_{p,\nu}^{p-1}|x|^{-2} \leq \nu^2|x|^{-2}.$$

Hence the corresponding to (4.4) quadratic form is coercive on the Sobolev space $D_0^1(\mathbb{R}^N \setminus K)$. Moreover, from Lemma 2.6 we conclude that given a large $R > 0$ there exists $m \in (0, \nu^2]$ such that

$$pU_*^{p-1}(x) \geq m|x|^{-2} \quad (|x| > R).$$

Then a standard application of the comparison principle for Hardy operators (cf. Lemma 2.5 and [12, Lemma A.8]) implies the two-sided bound

$$c|x|^{\alpha'_-} \leq h_\psi \leq C|x|^{\alpha_*^*} \quad (|x| > R),$$

where α_*^* is the smallest root of

$$-(\alpha + \nu_* - \nu)(\alpha + \nu_* + \nu) = pC_{p,\nu}^{p-1}$$

and α'_- is the smallest root of

$$-(\alpha + \nu_* - \nu)(\alpha + \nu_* + \nu) = m.$$

Note that $0 < m \leq pC_{p,\nu}^{p-1} \leq \nu^2$, so both equations have real roots and

$$-\nu_* - \nu < \alpha'_- \leq \alpha_*^* < -\nu_* < -\frac{2}{p-1}.$$

Set

$$\underline{U} := U_* - h_\psi.$$

Then

$$\lim_{|x| \rightarrow \infty} \frac{\underline{U}(x)}{U_*(x)} = 1,$$

and by convexity a direct computation shows

$$-\Delta \underline{U} = U_*^p - pU_*^{p-1}h_\psi \leq (U_* - h_\psi)^p = \underline{U}^p \quad \text{in } \mathbb{R}^N \setminus K,$$

that is \underline{U} is the required sub-solution. In addition,

$$\lim_{|x| \rightarrow \infty} \frac{U_*(x) - \underline{U}(x)}{|x|^{-\nu_*}} = \lim_{|x| \rightarrow \infty} \frac{h_\psi(x)}{|x|^{-\nu_*}} = 0,$$

which implies (4.3).

Supersolution \overline{U} . Let $\eta_\psi > 0$ be the minimal solution to the problem

$$-\Delta \eta + \frac{\nu^2 - \nu_*^2}{|x|^2} \eta = U_*^{p-1} \eta \quad \text{in } \mathbb{R}^N \setminus K, \quad \eta = U_* - \psi \quad \text{on } \partial K.$$

Note that $U_*^{p-1} \leq pU_*^{p-1}$. Hence, solution η_ψ exist simply because (4.5) applies. Moreover, a comparison argument similar to the ones above shows that

$$0 < \eta_\psi < h_\psi \quad \text{in } \mathbb{R}^N \setminus K.$$

Define

$$\overline{U} := U_* - \eta_\psi.$$

Then

$$\lim_{|x| \rightarrow \infty} \frac{\overline{U}(x)}{U_*(x)} = 1,$$

and

$$-\Delta \overline{U} = U_*^{p-1}(U_* - \eta_\psi) \geq (U_* - \eta_\psi)^{p-1}(U_* - \eta_\psi) = \overline{U}^p \quad \text{in } \mathbb{R}^N \setminus K,$$

that is \overline{U} is the required super-solution. \square

The next result shows that under suitable assumptions on the boundary data the exterior problem (4.2) admits a continuum of distinct slow decay positive solution, which in a certain sense could be interpreted as a perturbation of the family of slow decay solutions (U_λ) constructed in Theorem 3.1.

Corollary 4.6. *Let $p > p_S$, $\nu > 0$ and $pC_{p,\nu}^{p-1} \leq \nu^2$. Then for every $\psi \in C(\partial K)$ such that*

$$(4.6) \quad 0 \leq \psi(x) < U_\infty(x) \quad \text{on } \partial K,$$

problem (4.2) admits a continuum of distinct positive slow decay solutions.

Proof. Consider the family of slow decay solution $(U_\lambda)_{\lambda > 0}$, constructed in Theorem 3.1. In view of (4.6) there exists $\lambda_\psi > 0$ such that for all $\lambda > \lambda_\psi$

$$0 \leq \psi(x) < U_\lambda(x) < U_\infty(x) \quad \text{on } \partial K.$$

Let $\lambda_1 \in (\lambda_\psi, \infty]$. In Theorem 4.5, choose $U_* := U_{\lambda_1}$ and note that in view of (4.3) and (3.4) the solution $U_{\lambda_1}^\psi$ given by Theorem 4.5 is distinct with U_λ for any $\lambda > \lambda_\psi$, or with $U_{\lambda_2}^\psi$ for any other $\lambda_2 > \lambda_\psi$. In such a way we have obtained a family of distinct slow decay solutions $(U_\lambda^\psi)_{\lambda \in (\lambda_\psi, \infty]}$. \square

Remark 4.7. In particular, if $p > p_S$ and $pC_{p,\nu}^{p-1} \leq \nu^2$ then the problem

$$(4.7) \quad -\Delta u + \frac{\nu^2 - \nu_*^2}{|x|^2} u = u^p \quad \text{in } \mathbb{R}^N \setminus K, \quad u = 0 \quad \text{on } \partial K,$$

admits a continuum of distinct positive slow decay solutions $(U_\lambda^0)_{\lambda \in (0, \infty]}$. This partially extends the result in [4, Theorem 1]), obtained in the pure Laplacian case $\nu = \nu_*$. Note however that in [4] the existence of a continuum of slow decay solutions was proved for the whole range of exponents $p > p_S$, including the most challenging unstable regime $pC_{p,\nu_*}^{p-1} > \nu_*^2$. The techniques in [4] (see also a survey [5]) are based on linearization and perturbation arguments combined with a sophisticated machinery of harmonic expansions. Such considerations go beyond the scope of the present work.

Remark 4.8. In the pure Laplacian case $\nu = \nu_*$ it is known that if K is starshaped with respect to infinity then (4.7) has no positive solutions in the subcritical range $1 < p \leq p_S$, see [15, Theorem 2]. This suggests that the nonuniqueness statement of Corollary 4.6 can not be extended beyond the supercritical range of exponents.

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